On the Electrodynamics of Moving Bodies

IT IS WELL KNOWN that Maxwell's electrodynamics—as usually understood at present—when applied to moving bodies, leads to asymmetries that do not seem to be inherent in the phenomena. Take, for example, the electrodynamic interaction between a magnet and a conductor. The observable phenomenon here depends only on the relative motion of conductor and magnet, whereas the customary view draws a sharp distinction between the two cases, in which either the one or the other of the two bodies is in motion. For if the magnet is in motion and the conductor is at rest, an electric field with a definite energy value results in the vicinity of the magnet that produces a current wherever parts of the conductor are located. But if the magnet is at rest while the conductor is moving, no electric field results in the vicinity of the magnet, but rather an electromotive force in the conductor, to which no energy per se corresponds, but which, assuming an equality of relative motion in the two cases, gives rise to electric currents of the same magnitude and the same course as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to detect a motion of the earth relative to the "light medium," lead to the conjecture that not only the phenomena of mechanics but also those of electrodynamics have no properties that correspond to the concept of absolute rest. Rather, the same laws of electrodynamics and optics will be valid[1] for all coordinate systems in which the equations of mechanics hold, as has already been shown for quantities of the first order. We shall raise this conjecture (whose content will hereafter be called "the principle of relativity") to the status of a postulate and shall also introduce another postulate, which is only seemingly incompatible with it, namely that light always propagates in empty space with a definite velocity V that is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent electrodynamics of moving bodies based on Maxwell's theory for bodies at rest. The introduction of a "light ether" will prove to be superfluous, inasmuch as the view to be developed here will not require a "space at absolute rest" endowed with special properties, nor assign a velocity vector to a point of empty space where electromagnetic processes are taking place.

Like all electrodynamics, the theory to be developed here is based on the kinematics of a rigid body, since the assertions of any such theory have to do with the relations among rigid bodies (coordinate systems), clocks, and electromagnetic processes. Insufficient regard for this circumstance is at the root of the difficulties with which the electrodynamics of moving bodies currently has to contend.

A. Kinematic Part

1. Definition of Simultaneity

Consider a coordinate system in which Newton's mechanical equations are valid. To distinguish this system verbally from those to be introduced later, and to make our presentation more precise, we will call it the "rest system."

If a particle is at rest relative to this coordinate system, its position relative to the latter can be determined by means of rigid measuring rods using the methods of Euclidean geometry and expressed in Cartesian coordinates.

If we want to describe the *motion* of a particle, we give the values of its coordinates as functions of time. However, we must keep in mind that a mathematical description of this kind only has physical meaning if we are already clear as to what we understand here by "time." We have to bear in mind that all our judgments involving time are always judgments about *simultaneous events*. If, for example, I say that "the train arrives here at 7 o'clock," that means, more or less, "the pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events."[1]

It might seem that all difficulties involved in the definition of "time" could be overcome by my substituting "position of the small hand of my watch" for "time." Such a definition is indeed sufficient if a time is to be defined exclusively for the place at which the watch is located; but the definition is no longer satisfactory when series of events occurring at different locations have to be linked temporally, or—what

[1]We shall not discuss here the imprecision inherent in the concept of simultaneity of two events taking place at (approximately) the same location, which can be removed only by an abstraction.

amounts to the same thing—when events occurring at places remote from the clock have to be evaluated temporally.

To be sure, we could content ourselves with evaluating the time of events by stationing an observer with a clock at the origin of the coordinates who assigns to an event to be evaluated the corresponding position of the hands of the clock when a light signal from that event reaches him through empty space. However, we know from experience that such a coordination has the drawback of not being independent of the position of the observer with the clock. We reach a far more practical arrangement by the following argument.

If there is a clock at point A in space, then an observer located at A can evaluate the time of events in the immediate vicinity of A by finding the positions of the hands of the clock that are simultaneous with these events. If there is another clock at point B that in all respects resembles the one at A, then the time of events in the immediate vicinity of B can be evaluated by an observer at B. But it is not possible to compare the time of an event at A with one at B without a further stipulation. So far we have defined only an "A-time" and a "B-time," but not a common "time" for A and B. The latter can now be determined by establishing *by definition* that the "time" required for light to travel from A to B is equal to the "time" it requires to travel from B to A. For, suppose a ray of light leaves from A for B at "A-time" t_A, is reflected from B toward A at "B-time" t_B, and arrives back at A at "A-time" t'_A. The two clocks are synchronous by definition if

$$t_B - t_A = t'_A - t_B.$$

We assume that it is possible for this definition of synchronism to be free of contradictions, and to be so for arbitrarily many points, and therefore that the following relations are generally valid:

1. If the clock at B runs synchronously with the clock at A, the clock at A runs synchronously with the clock at B.
2. If the clock at A runs synchronously with the clock at B as well as with the clock at C, then the clocks at B and C also run synchronously relative to each other.

By means of certain (imagined) physical experiments, we have established what is to be understood by synchronous clocks at rest relative to each other and located at different places, and thereby obviously arrived at definitions of "synchronous" and "time." The "time" of an event is the reading obtained simultaneously from a clock at rest that is located at the place of the event, which for all time determinations runs synchronously with a specified clock at rest, and indeed with the specified clock.

Based on experience, we further stipulate that the quantity

$$\frac{2\overline{AB}}{t'_A - t_A} = V$$

be a universal constant (the velocity of light in empty space).

It is essential that we have defined time by means of clocks at rest in the rest system; because the time just defined is related to the system at rest, we call it "the time of the rest system."

2. *On the Relativity of Lengths and Times*

The following considerations are based on the principle of relativity and the principle of the constancy of the velocity of light. We define these two principles as follows:

1. If two coordinate systems are in uniform parallel translational motion relative to each other, the laws according to which the states of a physical system change do not depend on which of the two systems these changes are related to.
2. Every light ray moves in the "rest" coordinate system with a fixed velocity V, independently of whether this ray of light is emitted by a body at rest or in motion. Hence,

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}},$$

where "time interval" should be understood in the sense of the definition given in section 1.

Take a rigid rod at rest; let its length, measured by a measuring rod that is also at rest, be l. Now imagine the axis of the rod placed along the X-axis of the rest coordinate system, and the rod then set into uniform parallel translational motion (with velocity v) along the X-axis in the direction of increasing x. We now inquire about the length of the *moving* rod, which we imagine to be ascertained by the following two operations:

a. The observer moves together with the aforementioned measuring rod and the rigid rod to be measured, and measures the length of the rod by laying out the measuring rod in the same way as if the rod to be measured, the observer, and the measuring rod were all at rest.
b. Using clocks at rest and synchronous in the rest system as outlined in section 1, the observer determines at which points of the rest system the beginning and end of the rod to be measured are located at some given time t. The distance between these two points, measured with the rod used before—but now at rest—is also a length that we can call the "length of the rod."

According to the principle of relativity, the length determined by operation (a), which we shall call "the length of the rod in the moving system," must equal the length l of the rod at rest.

The length determined using operation (b), which we shall call "the length of the (moving) rod in the rest system," will be determined on the basis of our two principles, and we shall find that it differs from l.

Current kinematics tacitly assumes that the lengths determined by the above two operations are exactly equal to each other, or, in other words, that at the time t a moving rigid body is totally replaceable, in geometric respects, by the *same* body when it is *at rest* in a particular position.

Further, we imagine the two ends (A and B) of the rod equipped with clocks that are synchronous with the clocks of the rest system, i.e., whose readings always correspond to the "time of the system at rest" at the locations the clocks happen to occupy; hence, these clocks are "synchronous in the rest system."

We further imagine that each clock has an observer comoving with it, and that these observers apply to the two clocks the criterion for the synchronous rate of two clocks formulated in section 1. Let a ray of light start out from A at time[2] t_A; it is reflected from B at time t_B, and arrives back at A at time t'_A. Taking into account the principle of the constancy of the velocity of light, we find that

$$t_B - t_A = \frac{r_{AB}}{V - v}$$

[2] "Time" here means both "time of the system at rest" and "the position of the hands of the moving clock located at the place in question."

and

$$t'_A - t_B = \frac{r_{AB}}{V + v},$$

where r_{AB} denotes the length of the moving rod, measured in the rest system. Observers co-moving with the rod would thus find that the two clocks do not run synchronously, while observers in the system at rest would declare them to be running synchronously.

Thus we see that we cannot ascribe *absolute* meaning to the concept of simultaneity; instead, two events that are simultaneous when observed from some particular coordinate system can no longer be considered simultaneous when observed from a system that is moving relative to that system.

3. Theory of Transformations of Coordinate and Time from the Rest System to a System in Uniform Translational Motion Relative to It

Let there be two coordinate systems in the "rest" space, i.e., two systems of three mutually perpendicular rigid material lines originating from one point. Let the X-axes of the two systems coincide, and their Y- and Z-axes be respectively parallel. Each system shall be supplied with a rigid measuring rod and a number of clocks, and let both measuring rods and all the clocks of the two systems be exactly alike.

Now, put the origin of one of the two systems, say k, in a state of motion with (constant) velocity v in the direction of increasing x of the other system (K), which remains at rest; and let this new velocity be imparted to k's coordinate axes, its corresponding measuring rod, and its clocks. To each time t of the rest system K, there corresponds a definite location of the axes of the moving system. For reasons of symmetry we are justified to assume that the motion of k can be such that at time t ("t" always denotes a time of the rest system) the axes of the moving system are parallel to the axes of the rest system.

We now imagine space to be measured out from both the rest system K using the measuring rod at rest, and from the moving system k using the measuring rod moving along with it, and that coordinates x, y, z and ξ, η, ζ respectively are obtained in this way. Further, by means of the clocks at rest in the rest system, and using light signals as described in section 1, we determine the time t of the rest system for all the points where there are clocks. In a similar manner, by again applying the method of light signals described in section 1, we determine the time τ of the moving system, for all points of this moving system at which there are clocks at rest relative to this system.

To every set of values x, y, z, t which completely determines the place and time of an event in the rest system, there corresponds a set of values ξ, η, ζ, τ that fixes this event relative to the system k, and the problem to be solved now is to find the system of equations that connects these quantities.

First of all, it is clear that these equations must be *linear* because of the properties of homogeneity that we attribute to space and time.

If we put $x' = x - vt$, then it is clear that a point at rest in the system k has a definite, time-independent set of values x', y, z belonging to it. We first determine τ as a function of x', y, z, and t. To this end, we must express in equations that τ is in fact the aggregate of readings of clocks at rest in system k, synchronized according to the rule given in section 1.

Suppose that at time τ_0, a light ray is sent from the origin of the system k along the X-axis to x' and reflected from there toward the origin at time τ_1, arriving there at time τ_2; we then must have

$$\frac{1}{2}(\tau_0 + \tau_2) = \tau_1,$$

or, including the arguments of the function τ and applying the principle of the constancy of the velocity of light in the rest system,

$$\frac{1}{2}\left[\tau(0, 0, 0, t) + \tau\left(0, 0, 0, \left\{t + \frac{x'}{V - v} + \frac{x'}{V + v}\right\}\right)\right]$$
$$= \tau\left(x', 0, 0, t + \frac{x'}{V - v}\right).$$

From this we get, letting x' be infinitesimally small,

$$\frac{1}{2}\left(\frac{1}{V - v} + \frac{1}{V + v}\right)\frac{\partial \tau}{\partial t} = \frac{\partial \tau}{\partial x'} + \frac{1}{V - v}\frac{\partial \tau}{\partial t},$$

or

$$\frac{\partial \tau}{\partial x'} + \frac{v}{V^2 - v^2}\,\frac{\partial \tau}{\partial t} = 0.$$

It should be noted that, instead of the coordinate origin, we could have chosen any other point as the origin of the light ray, and therefore the equation just derived holds for all values of x', y, z.

Analogous reasoning—applied to the H[2] and Z axes—yields, remembering that light always propagates along these axes with the velocity $\sqrt{V^2 - v^2}$ when observed from the rest system,

$$\frac{\partial \tau}{\partial y} = 0$$

$$\frac{\partial \tau}{\partial z} = 0.$$

These equations yield, since τ is a *linear* function,

$$\tau = a\left(t - \frac{v}{V^2 - v^2} x'\right),$$

where a is a function $\varphi(v)$ as yet unknown, and where we assume for brevity that at the origin of k we have $t = 0$ when $\tau = 0$.

Using this result, we can easily determine the quantities ξ, η, ζ by expressing in equations that (as demanded by the principle of the constancy of the velocity of light in conjunction with the principle of relativity) light also propagates with velocity V when measured in the moving system. For a light ray emitted at time $\tau = 0$ in the direction of increasing ξ, we have

$$\xi = V\tau,$$

or

$$\xi = aV\left(t - \frac{v}{V^2 - v^2} x'\right).$$

But as measured in the rest system, the light ray propagates with velocity $V - v$ relative to the origin of k, so that

$$\frac{x'}{V - v} = t.$$

Substituting this value of t in the equation for ξ, we obtain

$$\xi = a \frac{V^2}{V^2 - v^2} x'.$$

Analogously, by considering light rays moving along the two other axes, we get

$$\eta = V\tau = aV\left(t - \frac{v}{V^2 - v^2} x'\right),$$

where

$$\frac{y}{\sqrt{V^2 - v^2}} = t; \qquad x' = 0;$$

hence

$$\eta = a \frac{V}{\sqrt{V^2 - v^2}} \; y$$

and

$$\zeta = a \frac{V}{\sqrt{V^2 - v^2}} \; z.$$

If we substitute for x' its value, we obtain

$$\tau = \varphi(v)\,\beta\Big(t - \frac{v}{V^2}\,x\Big),$$

$$\xi = \varphi(v)\,\beta(x - vt),$$

$$\eta = \varphi(v)y,$$

$$\zeta = \varphi(v)z,$$

where

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}$$

and φ is an as yet unknown function of v. If no assumptions are made regarding the initial position of the moving system and the zero point of τ, then a constant must be added to the right-hand sides of these equations.

Now we have to prove that, measured in the moving system, every light ray propagates with the velocity V, if it does so, as we have assumed, in the rest system; for we have not yet proved that the principle of the constancy of the velocity of light is compatible with the relativity principle.

Suppose that at time $t = \tau = 0$ a spherical wave is emitted from the coordinate origin, which at that time is common to both systems, and that this wave propagates in the system K with the velocity V. Hence, if (x, y, z) is a point reached by this wave, we have

$$x^2 + y^2 + z^2 = V^2 t^2.$$

We transform this equation using our transformation equations and, after a simple calculation, obtain

$$\xi^2 + \eta^2 + \zeta^2 = V^2 \tau^2.$$

Thus, our wave is also a spherical wave with propagation velocity V when it is observed in the moving system. This proves that our two fundamental principles are compatible.[3]

The transformation equations we have derived also contain an unknown function φ of v, which we now wish to determine.

To this end we introduce a third coordinate system K', which, relative to the system k, is in parallel-translational motion, parallel to the axis Ξ,[4] such that its origin moves along the Ξ-axis with velocity $-v$. Let all three coordinate origins coincide at time $t = 0$, and let the time t' of system K' equal zero at $t = x = y = z = 0$. We denote the coordinates measured in the system K' by x', y', z' and, by twofold application of our transformation equations, we get

$$\begin{aligned}
t' &= \varphi(-v)\beta(-v)\left\{\tau + \frac{v}{V^2}\xi\right\} &&= \varphi(v)\varphi(-v)t,\\
x' &= \varphi(-v)\beta(-v)\{\xi + v\tau\} &&= \varphi(v)\varphi(-v)x,\\
y' &= \varphi(-v)\eta &&= \varphi(v)\varphi(-v)y,\\
z' &= \varphi(-v)\zeta &&= \varphi(v)\varphi(-v)z.
\end{aligned}$$

Since the relations between x', y', z' and x, y, z do not contain the time t, the systems K and K' are at rest relative to each other, and it is clear that the transformation from K to K' must be the identity transformation. Hence,

$$\varphi(v)\varphi(-v) = 1.$$

Let us now explore the meaning of $\varphi(v)$. We shall focus on that portion of the H-axis of the system k that lies between $\xi = 0$, $\eta = 0$, $\zeta = 0$, and $\xi = 0$, $\eta = l$, $\zeta = 0$. This portion of the H-axis is a rod that, relative to the system K, moves perpendicular to its axis with a velocity v and its ends have coordinates in K:

$$x_1 = vt, \quad y_1 = \frac{l}{\varphi(v)}, \quad z_1 = 0$$

and

$$x_2 = vt, \quad y_2 = 0, \quad z_2 = 0.$$

The length of the rod, measured in K, is thus $l/\varphi(v)$; this gives us the meaning of the function φ. For reasons of symmetry, it is now evident that the length of a rod measured in the rest system and moving perpendicular to its axis can depend only on its velocity and not on the direction and sense of its motion. Thus, the length of the moving rod measured in the rest system does not change if v is replaced by $-v$. From this we conclude:

$$\frac{l}{\varphi(v)} = \frac{l}{\varphi(-v)},$$

or

$$\varphi(v) = \varphi(-v).$$

From this relation and the one found earlier it follows that $\varphi(v) = 1$, so that the transformation equations obtained become

$$\tau = \beta\left(t - \frac{v}{V^2}\,x\right),$$

$$\xi = \beta(x - vt),$$

$$\eta = y,$$

$$\zeta = z,$$

where

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.$$

4. The Physical Meaning of the Equations Obtained as Concerns Moving Rigid Bodies and Moving Clocks

We consider a rigid sphere[3] of radius R that is at rest relative to the moving system k and whose center lies at the origin of k. The equation of the surface of this sphere, which moves with velocity v relative to the system k, is

$$\xi^2 + \eta^2 + \zeta^2 = R^2.$$

Expressed in terms of x, y, z, the equation of this surface at time $t = 0$ is

$$\frac{x^2}{\left(\sqrt{1 - \left(\frac{v}{V}\right)^2}\right)^2} + y^2 + z^2 = R^2.$$

A rigid body that has a spherical shape when measured at rest has, when in motion—considered from the rest

[3] I.e., a body that has a spherical shape when examined at rest.

system—the shape of an ellipsoid of revolution with axes

$$R\sqrt{1-\left(\frac{v}{V}\right)^2},\ R,\ R.$$

Thus, while the Y and Z dimensions of the sphere (and hence also of every rigid body, whatever its shape) do not appear to be altered by motion, the X dimension appears to be contracted in the ratio $1 : \sqrt{1-(v/V)^2}$, thus the greater the value of v, the greater the contraction. For $v = V$, all moving objects—considered from the "rest" system—shrink into plane structures. For superluminal velocities our considerations become meaningless; as we shall see from later considerations, in our theory the velocity of light physically plays the role of infinitely great velocities.

It is clear that the same results apply for bodies at rest in the "rest" system when considered from a uniformly moving system.

We further imagine one of the clocks that is able to indicate time t when at rest relative to the rest system and time τ when at rest relative to the moving system to be placed at the origin of k and set such that it indicates the time τ. What is the rate of this clock when considered from the rest system?

The quantities x, t, and τ that refer to the position of this clock obviously satisfy the equations

$$\tau = \frac{1}{\sqrt{1-\left(\frac{v}{V}\right)^2}}\left(t - \frac{v}{V^2}x\right)$$

and

$$x = vt.$$

We thus have

$$\tau = t\sqrt{1 - \left(\frac{v}{V}\right)^2} = t - \left(1 - \sqrt{1 - \left(\frac{v}{V}\right)^2}\right)t,$$

from which it follows that the reading of the clock considered from the rest system lags behind each second by $(1 - \sqrt{1 - (v/V)^2})$ sec or, up to quantities of the fourth and higher order, by $\frac{1}{2}(v/V)^2$ sec.

This yields the following peculiar consequence: If at the points A and B of K there are clocks at rest that, considered from the rest system, are running synchronously, and if the clock at A is transported to B along the connecting line with velocity v, then upon arrival of this clock at B the two clocks will no longer be running synchronously; instead, the clock that has been transported from A to B will lag $\frac{1}{2}tv^2/V^2$ sec (up to quantities of the fourth and higher orders) behind the clock that has been in B from the outset, where t is the time needed by the clock to travel from A to B.

We see at once that this result holds even when the clock moves from A to B along any arbitrary polygonal line, and even when the points A and B coincide.[5]

If we assume that the result proved for a polygonal line holds also for a continuously curved line, then we arrive at the following result: If there are two synchronously running clocks at A, and one of them is moved along a closed curve with constant velocity until it has returned to A, which takes, say, t sec, then, on its arrival at A, this clock will lag $\frac{1}{2}t(v/V)^2$ sec behind the clock that has not been moved. From this we conclude that a balance-wheel clock[6] located at the Earth's equator must, under otherwise identical conditions, run more slowly by a very small amount than an absolutely identical clock located at one of the Earth's poles.

5. *The Addition Theorem for Velocities*

In the system k moving with velocity v along the X-axis of the system K, let a point move in accordance with the equations

$$\xi = w_\xi \tau,$$
$$\eta = w_\eta \tau,$$
$$\zeta = 0,$$

where w_ξ and w_η denote constants.

We seek the motion of the point relative to the system K. Introducing the quantities x, y, z, t into the equations of motion of the point by means of the transformation equations derived in section 3, we obtain

$$x = \frac{w_\xi + v}{1 + \frac{v w_\xi}{V^2}} t,$$

$$y = \frac{\sqrt{1 - \left(\frac{v}{V}\right)^2}}{1 + \frac{v w_\xi}{V^2}} w_\eta t,$$

$$z = 0.$$

Thus, according to our theory, the vector addition for velocities holds only to first approximation. Let

$$U^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

$$w^2 = w_\xi^2 + w_\eta^2$$

and

$$\alpha = \arctan \frac{w_y}{w_x};\text{[7]}$$

α is then to be considered as the angle between the velocities v and w. After a simple calculation we obtain

$$U = \frac{\sqrt{(v^2 + w^2 + 2vw\cos\alpha) - \left(\frac{vw\sin\alpha}{V}\right)^2}}{1 + \frac{vw\cos\alpha}{V^2}}.$$

It is worth noting that v and w enter into the expression for the resultant velocity in a symmetrical manner. If w also has the direction of the X-axis (Ξ-axis), we get

$$U = \frac{v + w}{1 + \frac{vw}{V^2}}.$$

It follows from this equation that the composition of two velocities that are smaller than V always results in a velocity that is smaller than V. For if we set $v = V - \kappa$, and $w = V - \lambda$, where κ and λ are positive and smaller than V, then

$$U = V\,\frac{2V - \kappa - \lambda}{2V - \kappa - \lambda + \frac{\kappa\lambda}{V}} < V.$$

It also follows that the velocity of light V cannot be changed by compounding it with a "subluminal velocity." For this case we get

$$U = \frac{V + w}{1 + \frac{w}{V}} = V.$$

In the case where v and w have the same direction, the formula for U could also have been obtained by compounding two transformations according to section 3. If in addition to the systems K and k, occurring in section 3, we introduce a third coordinate system k', which moves parallel to k and whose origin moves with velocity w along the Ξ-axis, we obtain equations between the quantities x, y, z, t and the corresponding quantities of k' that differ from those found in section 3 only insofar as "v" is replaced by the quantity

$$\frac{v+w}{1+\frac{vw}{V^2}};$$

from this we see that such parallel transformations form a group—as indeed they must.

We have now derived the required laws of the kinematics corresponding to our two principles, and proceed to their application to electrodynamics.

B. Electrodynamic Part

6. Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Arising Due to Motion in a Magnetic Field

Let the Maxwell-Hertz equations for empty space be valid for the rest system K, so that we have

$$\begin{aligned}
\frac{1}{V}\frac{\partial X}{\partial t} &= \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, & \frac{1}{V}\frac{\partial L}{\partial t} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y},\\
\frac{1}{V}\frac{\partial Y}{\partial t} &= \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, & \frac{1}{V}\frac{\partial M}{\partial t} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z},\\
\frac{1}{V}\frac{\partial Z}{\partial t} &= \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}, & \frac{1}{V}\frac{\partial N}{\partial t} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x},
\end{aligned}$$

where (X, Y, Z) denotes the electric force vector and (L, M, N) the magnetic force vector.

If we apply the transformations derived in section 3 to these equations, in order to relate the electromagnetic processes to the coordinate system moving with velocity v introduced there, we obtain the following equations:

$$\frac{1}{V}\frac{\partial X}{\partial \tau} = \frac{\partial \beta\left(N - \frac{v}{V}Y\right)}{\partial \eta} - \frac{\partial \beta\left(M + \frac{v}{V}Z\right)}{\partial \zeta},$$

$$\frac{1}{V}\frac{\partial \beta\left(Y - \frac{v}{V}N\right)}{\partial \tau} = \frac{\partial L}{\partial \zeta} - \frac{\partial \beta\left(N - \frac{v}{V}Y\right)}{\partial \xi},$$

$$\frac{1}{V}\frac{\partial \beta\left(Z + \frac{v}{V}M\right)}{\partial \tau} = \frac{\partial \beta\left(M + \frac{v}{V}Z\right)}{\partial \xi} - \frac{\partial L}{\partial \eta},$$

$$\frac{1}{V}\frac{\partial L}{\partial \tau} = \frac{\partial \beta\left(Y - \frac{v}{V}N\right)}{\partial \zeta} - \frac{\partial \beta\left(Z + \frac{v}{V}M\right)}{\partial \eta},$$

$$\frac{1}{V}\frac{\partial \beta\left(M + \frac{v}{V}Z\right)}{\partial \tau} = \frac{\partial \beta\left(Z + \frac{v}{V}M\right)}{\partial \xi} - \frac{\partial X}{\partial \zeta},$$

$$\frac{1}{V}\frac{\partial \beta\left(N - \frac{v}{V}Y\right)}{\partial \tau} = \frac{\partial X}{\partial \eta} - \frac{\partial \beta\left(Y - \frac{v}{V}N\right)}{\partial \xi},$$

where

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.$$

The relativity principle requires that the Maxwell-Hertz equations for empty space also be valid in the system k if they are valid in the system K, i.e, that the electric and magnetic force vectors—(X', Y', Z') and (L', M', N')—of the moving system k, which are defined in this system by their ponderomotive effects on electric and magnetic charges, respectively, satisfy the equations

$$\frac{1}{V}\frac{\partial X'}{\partial \tau} = \frac{\partial N'}{\partial \eta} - \frac{\partial M'}{\partial \zeta}, \qquad \frac{1}{V}\frac{\partial L'}{\partial \tau} = \frac{\partial Y'}{\partial \zeta} - \frac{\partial Z'}{\partial \eta},$$

$$\frac{1}{V}\frac{\partial Y'}{\partial \tau} = \frac{\partial L'}{\partial \zeta} - \frac{\partial N'}{\partial \xi}, \qquad \frac{1}{V}\frac{\partial M'}{\partial \tau} = \frac{\partial Z'}{\partial \xi} - \frac{\partial X'}{\partial \zeta},$$

$$\frac{1}{V}\frac{\partial Z'}{\partial \tau} = \frac{\partial M'}{\partial \xi} - \frac{\partial L'}{\partial \eta}, \qquad \frac{1}{V}\frac{\partial N'}{\partial \tau} = \frac{\partial X'}{\partial \eta} - \frac{\partial Y'}{\partial \xi}.$$

Obviously, the two systems of equations found for the system k must express exactly the same thing, since both systems of equations are equivalent to the Maxwell-Hertz equations for the system K. Further, since the equations for the two systems are in agreement apart from the symbols representing the vectors, it follows that the functions occurring in the systems of equations at corresponding places must agree up to a factor $\psi(v)$, common to all functions of one of the systems of equations and independent of ξ, η, ζ, and τ, but possibly depending on v. Thus we have the relations:

$$X' = \psi(v)X, \qquad L' = \psi(v)L,$$

$$Y' = \psi(v)\beta\left(Y - \frac{v}{V}N\right), \qquad M' = \psi(v)\beta\left(M + \frac{v}{V}Z\right),$$

$$Z' = \psi(v)\beta\left(Z + \frac{v}{V}M\right), \qquad N' = \psi(v)\beta\left(N - \frac{v}{V}Y\right).$$

If we now invert this system of equations, first by solving the equations just obtained, and second by applying to the equations the inverse transformation (from k to K), which is characterized by the velocity $-v$, we get, taking into account that both systems of equations so obtained must be identical,

$$\varphi(v) \cdot \varphi(-v) = 1.$$

Further, it follows for reasons of symmetry[4] that

$$\varphi(v) = \varphi(-v);$$

[4] If, e.g., $X = Y = Z = L = M = 0$ and $N \neq 0$, then it is clear for reasons of symmetry that if v changes its sign without changing its numerical value, then Y' must also change its sign without changing its numerical value.

thus

$$\varphi(v) = 1,$$

and our equations take the form

$$X' = X, \qquad L' = L,$$
$$Y' = \beta\left(Y - \frac{v}{V}N\right), \qquad M' = \beta\left(M + \frac{v}{V}Z\right),$$
$$Z' = \beta\left(Z + \frac{v}{V}M\right), \qquad N' = \beta\left(N - \frac{v}{V}Y\right).$$

By way of interpreting these equations, we note the following remarks: Imagine a pointlike electric charge, whose magnitude measured in the rest system is "unit," i.e., which, when at rest in the rest system exerts a force of 1 dyne on an equal charge at a distance of 1 cm. According to the principle of relativity this electric charge is also of "unit" magnitude if measured in the moving system. If this electric charge is at rest relative to the rest system, then by definition the vector (X, Y, Z) equals the force acting on it. If, on the other hand, this acting charge is at rest relative to the moving system (at least at the relevant instant), then the force acting on it measured in the moving system is equal to the vector (X', Y', Z'). Hence, the first three of the above equations can be expressed in words in the following two ways:

1. If a unit point electric charge moves in an electromagnetic field, there acts upon it, in addition to the electric force, an "electromotive force" that, neglecting terms multiplied by the second and higher powers of v/V, is equal to the vector product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old mode of expression.)
2. If a unit point electric charge moves in an electromagnetic field, the force acting on it equals the electric force at the

location of the unit charge that is obtained by transforming the field to a coordinate system at rest relative to the unit charge. (New mode of expression.)

Analogous remarks hold for "magnetomotive forces."[8] We can see that in the theory developed here, the electromotive force only plays the role of an auxiliary concept, which owes its introduction to the circumstance that the electric and magnetic forces do not have an existence independent of the state of motion of the coordinate system.

It is further clear that the asymmetry in the treatment of currents produced by the relative motion of a magnet and a conductor, mentioned in the introduction, disappears. Moreover, questions about the "site" of electrodynamic electromotive forces (unipolar machines) become pointless.

7. Theory of Doppler's Principle and of Aberration

In the system K and very far from the coordinate origin, let there be a source of electrodynamic waves that, in a part of space containing the coordinate origin, are represented with sufficient accuracy by the equations

$$\begin{aligned} X &= X_0 \sin\Phi, & L &= L_0 \sin\Phi, & & \\ Y &= Y_0 \sin\Phi, & M &= M_0 \sin\Phi, & \Phi &= \omega\left(t - \frac{ax+by+cz}{V}\right). \\ Z &= Z_0 \sin\Phi, & N &= N_0 \sin\Phi, & & \end{aligned}$$

Here (X_0, Y_0, Z_0) and (L_0, M_0, N_0) are the vectors determining the amplitude of the wave train, and a, b, c are the direction cosines of the norm to the waves.

We want to know the character of these waves when investigated by an observer at rest in the moving system k.

Applying the transformation equations for electric and magnetic forces found in section 6 and those for coordinates and time found in section 3, we immediately obtain:

$$X' = X_0 \sin \Phi', \qquad L' = L_0 \sin \Phi',$$

$$Y' = \beta\Big(Y_0 - \frac{v}{V}N_0\Big) \sin \Phi', \qquad M' = \beta\Big(M_0 + \frac{v}{V}Z_0\Big) \sin \Phi',$$

$$Z' = \beta\Big(Z_0 + \frac{v}{V}M_0\Big) \sin \Phi', \qquad N' = \beta\Big(N_0 - \frac{v}{V}Y_0\Big) \sin \Phi',$$

$$\Phi' = \omega'\left(\tau - \frac{a'\xi + b'\eta + c'\zeta}{V}\right),$$

where we have put

$$\omega' = \omega\beta\Big(1 + a\frac{v}{V}\Big),$$

$$a' = \frac{a - \frac{v}{V}}{1 - a\frac{v}{V}},$$

$$b' = \frac{b}{\beta\left(1 - a\frac{v}{V}\right)},$$

$$c' = \frac{c}{\beta\left(1 - a\frac{v}{V}\right)}.$$

From the equation for ω' it follows that if an observer moves with velocity v relative to an infinitely distant source of light of frequency ν, in such a way that the connecting line "light source–observer" forms an angle φ with the observer's velocity, where this velocity is relative to a coordinate system at rest relative to the light source, then ν', the frequency of the light perceived by the observer, is given by the equation

$$\nu' = \nu \frac{1 - \cos\varphi \frac{v}{V}}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.$$

This is Doppler's principle for arbitrary velocities. For $\varphi = 0$ the equation takes the simple form

$$\nu' = \nu \sqrt{\frac{1 - \frac{v}{V}}{1 + \frac{v}{V}}}.$$

We see that, contrary to the usual conception, when $v = -\infty$, then $\nu = \infty$.[9]

If φ' denotes the angle between the wave normal (the direction of the ray) in the moving system and the connecting line "light source–observer,"[10] the equation for α'[11] takes the form

$$\cos\varphi' = \frac{\cos\varphi - \frac{v}{V}}{1 - \frac{v}{V}\cos\varphi}.$$

This equation expresses the law of aberration in its most general form. If $\varphi = \pi/2$, the equation takes the simple form

$$\cos\varphi' = -\frac{v}{V}.$$

We still need to find the amplitude of the waves as it appears in the moving system. If A and A' denote the amplitude of electric or magnetic force in the rest system and moving system respectively, we get

$$A'^2 = A^2 \frac{\left(1 - \frac{v}{V}\cos\varphi\right)^2}{1 - \left(\frac{v}{V}\right)^2},$$

which for $\varphi = 0$ takes the simpler form:

$$A'^2 = A^2 \frac{1 - \frac{v}{V}}{1 + \frac{v}{V}}.$$

It follows from these results that to an observer approaching a light source with velocity V, this source would have to appear infinitely intense.

8. Transformation of the Energy of Light Rays. Theory of Radiation Pressure Exerted on Perfect Mirrors

Since $A^2/8\pi$ equals the energy of light per unit volume, according to the principle of relativity we have to consider $A'^2/8\pi$ as the light energy in the moving system. Hence A'^2/A^2 would be the ratio of the energy of a given light complex "measured in motion" and "measured at rest" if the volume of a light complex were the same measured in K and in k. However, this is not the case. If a, b, c are the direction cosines of the wave normal of the light in the rest system, then the surface elements of the spherical surface

$$(x - Vat)^2 + (y - Vbt)^2 + (z - Vct)^2 = R^2$$

moving with the velocity of light are not traversed by any energy; we may therefore say that this surface permanently encloses the same light complex. We investigate the quantity of energy enclosed by this surface considered from the system k, i.e., the energy of the light complex relative to the system k.

Considered in the moving system, the spherical surface is an ellipsoidal surface whose equation at time $\tau = 0$ is

$$\left(\beta\xi - a\beta\frac{v}{V}\xi\right)^2 + \left(\eta - b\beta\frac{v}{V}\xi\right)^2 + \left(\zeta - c\beta\frac{v}{V\xi}\right)^2 = R^2.$$

If S denotes the volume of the sphere and S' that of the ellipsoid, then a simple calculation shows that

$$\frac{S'}{S} = \frac{\sqrt{1 - \left(\frac{v}{V}\right)^2}}{1 - \frac{v}{V}\cos\varphi}.$$

If we call the energy of the light enclosed by this surface E when measured in the rest system and E' when measured in the moving system, we obtain

$$\frac{E'}{E} = \frac{\frac{A'^2}{8\pi}S'}{\frac{A^2}{8\pi}S} = \frac{1 - \frac{v}{V}\cos\varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}},$$

which, for $\varphi = 0$, simplifies to

$$\frac{E'}{E} = \sqrt{\frac{1 - \frac{v}{V}}{1 + \frac{v}{V}}}.$$

It is noteworthy that the energy and the frequency of a light complex vary with the observer's state of motion according to the same law.

Let the coordinate plane $\xi = 0$ be a completely reflecting surface, from which the plane waves considered in section 7 are reflected. We investigate the pressure of light exerted on the reflecting surface, and the direction, frequency, and intensity of the light after reflection.

Let the incident light be defined by the quantities A, $\cos\ \varphi$, and ν (relative to system K). Considered from k, the corresponding quantities are

$$A' = A\ \frac{1 - \frac{v}{V}\cos\varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}},$$

$$\cos\varphi' = \frac{\cos\varphi - \frac{v}{V}}{1 - \frac{v}{V}\cos\varphi},$$

$$\nu' = \nu\ \frac{1 - \frac{v}{V}\cos\varphi}{\sqrt{1 - \left(\frac{v}{V}\right)^2}}.$$

Referring the process to the system k, we get for the reflected light

$$A'' = A',$$

$$\cos\varphi'' = -\cos\varphi',$$

$$\nu'' = \nu'.$$

Finally, by transforming back to the rest system K, we get for the reflected light

$$A''' = A''\ \frac{1 + \frac{v}{V}\cos\varphi''}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} = A\ \frac{1 - 2\frac{v}{V}\cos\varphi + \left(\frac{v}{V}\right)^2}{1 - \left(\frac{v}{V}\right)^2},$$

$$\cos\varphi''' = \frac{\cos\varphi'' + \frac{v}{V}}{1 + \frac{v}{V}\cos\varphi''} = -\frac{\left(1 + \left(\frac{v}{V}\right)^2\right)\cos\varphi - 2\frac{v}{V}}{1 - 2\frac{v}{V}\cos\varphi + \left(\frac{v}{V}\right)^2},$$

$$\nu''' = \nu'' \frac{1 + \frac{v}{V}\cos\varphi''}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} = \nu \frac{1 - 2\frac{v}{V}\cos\varphi + \left(\frac{v}{V}\right)^2}{\left(1 - \frac{v}{V}\right)^2}.^{[12]}$$

The energy (measured in the rest system) that strikes a unit surface of the mirror per unit time is obviously $A^2/8\pi(V\cos\varphi - v)$. The energy leaving a unit surface of the mirror per unit time is $A'''^2/8\pi(-V\cos\varphi''' + v)$. According to the principle of energy conservation, the difference of these two expressions is the work done by light pressure per unit time. Equating this work to $P \cdot v$, where P is the pressure of light, we obtain

$$P = 2\,\frac{A^2}{8\pi}\frac{\left(\cos\varphi - \frac{v}{V}\right)^2}{1 - \left(\frac{v}{V}\right)^2}.$$

To first approximation, in agreement with experiment and with other theories, we get

$$P = 2\,\frac{A^2}{8\pi}\cos^2\varphi.$$

All problems in the optics of moving bodies can be solved by the method employed here. The essential point is that the electric and magnetic fields of light that is influenced by a moving body are transformed to a coordinate system that is at rest relative to that body. By this means, all problems in the optics of moving bodies are reduced to a series of problems in the optics of bodies at rest.

9. Transformation of the Maxwell-Hertz Equations When Convection Currents Are Taken into Account

We start from the equations

$$\frac{1}{V}\left\{u_x\rho + \frac{\partial X}{\partial t}\right\} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \qquad \frac{1}{V}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y},$$

$$\frac{1}{V}\left\{u_y\rho + \frac{\partial Y}{\partial t}\right\} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \qquad \frac{1}{V}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z},$$

$$\frac{1}{V}\left\{u_z\rho + \frac{\partial Z}{\partial t}\right\} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}, \qquad \frac{1}{V}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x},$$

where

$$\rho = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$$

denotes 4π times the charge density, and (u_x, u_y, u_z) the velocity vector of the charge. If the electric charges are conceived as permanently bound to small, rigid bodies (ions, electrons), then these equations constitute the electromagnetic foundation of Lorentz's electrodynamics and optics of moving bodies.

If, using the transformation equations presented in sections 3 and 6, we transform these equations, assumed to be valid in the system K, to the system k, we get the equations

$$\frac{1}{V}\left\{u_\xi\rho' + \frac{\partial X'}{\partial \tau}\right\} = \frac{\partial N'}{\partial \eta} - \frac{\partial M'}{\partial \zeta}, \qquad \frac{1}{V}\frac{\partial L'}{\partial \tau} = \frac{\partial Y'}{\partial \zeta} - \frac{\partial Z'}{\partial \eta},$$

$$\frac{1}{V}\left\{u_\eta\rho' + \frac{\partial Y'}{\partial \tau}\right\} = \frac{\partial L'}{\partial \zeta} - \frac{\partial N'}{\partial \xi}, \qquad \frac{1}{V}\frac{\partial M'}{\partial \tau} = \frac{\partial Z'}{\partial \xi} - \frac{\partial X'}{\partial \zeta},$$

$$\frac{1}{V}\left\{u_\zeta\rho' + \frac{\partial Z'}{\partial \tau}\right\} = \frac{\partial M'}{\partial \xi} - \frac{\partial L'}{\partial \eta}, \qquad \frac{1}{V}\frac{\partial N'}{\partial \tau} = \frac{\partial X'}{\partial \eta} - \frac{\partial Y'}{\partial \xi},$$

where

$$\frac{u_x - v}{1 - \frac{u_x v}{V^2}} = u_\xi,$$

$$\frac{u_y}{\beta\left(1 - \frac{u_x v}{V^2}\right)} = u_\eta,$$

$$\frac{u_z}{\beta\left(1 - \frac{u_x v}{V^2}\right)} = u_\zeta,$$

and

$$\rho' = \frac{\partial X'}{\partial \xi} + \frac{\partial Y'}{\partial \eta} + \frac{\partial Z'}{\partial \zeta} = \beta\left(1 - \frac{v u_x}{V^2}\right)\rho.$$

Since—as follows from the velocity addition theorem (sec. 5)—the vector (u_ξ, u_η, u_ζ) is actually the velocity of the electric charges measured in the system k, we have thus shown that, on the basis of our kinematic principles, the electrodynamic foundation of Lorentz's theory of the electrodynamics of moving bodies agrees with the principle of relativity.

Let me also briefly add that the following important proposition can easily be deduced from the equations we have derived: If an electrically charged body moves arbitrarily in space without altering its charge when observed from a coordinate system moving with the body, then its charge also remains constant when observed from the "rest" system K.

10. Dynamics of the (Slowly Accelerated) Electron

In an electromagnetic field let there move an electrically charged particle with charge ϵ (called an "electron" in what follows); we assume only the following about its law of motion:

If the electron is at rest at a particular moment, its motion during the next instant of time will occur according to the equations

$$\mu \frac{d^2x}{dt^2} = \epsilon X,$$

$$\mu \frac{d^2y}{dt^2} = \epsilon Y,$$

$$\mu \frac{d^2z}{dt^2} = \epsilon Z,$$

where x, y, z denote the coordinates of the electron and μ its mass as long as the electron moves slowly.

Further, let the electron's velocity at a certain moment be v. We investigate the law of motion of the electron during the immediately succeeding instant of time.

Without loss of generality, we may and shall assume that the electron is at the coordinate origin and moves with velocity v along the X-axis of the system K at the moment with which we are concerned. It is then obvious that at the given moment ($t = 0$), the electron is at rest relative to a coordinate system k moving with constant velocity v parallel to the X-axis.

From the above assumption combined with the relativity principle, it is clear that, considered from the system k, the electron will move during the immediately ensuing period of time (for small values of t) according to the equations

$$\mu \frac{d^2\xi}{d\tau^2} = \epsilon X',$$

$$\mu \frac{d^2\eta}{d\tau^2} = \epsilon Y',$$

$$\mu \frac{d^2\zeta}{d\tau^2} = \epsilon Z',$$

where the symbols ξ, η, ζ, τ, X', Y', Z' all refer to the system k. If we also stipulate that, for $t = x = y = z = 0$, $\tau = \xi = \eta = \zeta = 0$ shall also hold, then the transformation equations of sections 3 and 6 are applicable, so that we get

$$\tau = \beta\left(t - \frac{v}{V^2}x\right),$$

$$\xi = \beta(x - vt), \qquad X' = x,$$

$$\eta = y, \qquad Y' = \beta\left(Y - \frac{v}{V}N\right),$$

$$\zeta = z, \qquad Z' = \beta\left(Z + \frac{v}{V}M\right).$$

With the help of these equations we transform the above equations of motion from the system k to the system K, obtaining

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{\epsilon}{\mu}\frac{1}{\beta^3}X, \\ \frac{d^2y}{dt^2} &= \frac{\epsilon}{\mu}\frac{1}{\beta}\left(Y - \frac{v}{V}N\right), \\ \frac{d^2z}{dt^2} &= \frac{\epsilon}{\mu}\frac{1}{\beta}\left(Z + \frac{v}{V}M\right). \end{aligned} \tag{A}$$

Following the usual approach, we now investigate the "longitudinal" and "transverse" mass of the moving electron. We write equations (A) in the form

$$\mu\beta^3\frac{d^2x}{dt^2} = \epsilon X = \epsilon X',$$

$$\mu\beta^2\frac{d^2y}{dt^2} = \epsilon\beta\left(Y - \frac{v}{V}N\right) = \epsilon Y',$$

$$\mu\beta^2\frac{d^2z}{dt^2} = \epsilon\beta\left(Z + \frac{v}{V}M\right) = \epsilon Z',$$

and note first that $\epsilon X'$, $\epsilon Y'$, $\epsilon Z'$ are the components of the ponderomotive force acting on the electron, as considered in a moving system that, at this instant, is moving with the same velocity as the electron. (This force could be measured, for example, by a spring balance at rest in the latter system.) If we simply call[13] this force "the force acting on the electron," and preserve the equation

$$\text{Mass} \times \text{Acceleration} = \text{Force},$$

stipulating, in addition, that accelerations be measured in the rest system K, then the above equations lead to the definition:

$$\text{Longitudinal mass} = \frac{\mu}{\left(\sqrt{1-\left(\frac{v}{V}\right)^2}\right)^3},$$

$$\text{Transverse mass} = \frac{\mu}{1-\left(\frac{v}{V}\right)^2}.$$

Of course, with a different definition of force and acceleration we would obtain different values for these masses; this shows that we must proceed very cautiously when comparing various theories of electron motion.

It should be noted that these results about mass are also valid for ponderable material points, because a ponderable material point can be made into an electron (in our sense of the word) by adding to it an *arbitrarily small* electric charge.

We now determine the kinetic energy of an electron. If an electron starts from the origin of the system K with an initial velocity 0 and continues to move along the X-axis under the influence of an electrostatic force X, it is clear that the energy it takes from the electrostatic field has the value

$\int \epsilon X dx$. Since the electron is supposed to accelerate slowly, and consequently cannot emit any energy in the form of radiation, the energy taken from the electrostatic field must be equated to the kinetic energy W of the electron. Bearing in mind that the first of equations (A) holds throughout the entire process of motion, we obtain

$$W = \int \epsilon X \, dx = \int_0^v \beta^3 v \, dv = \mu V^2 \left\{ \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}} - 1 \right\}.$$

Thus, W becomes infinitely large when $v = V$. As is the case for our previous results, superluminal velocities are not possible.

By virtue of the argument presented above, this expression for kinetic energy must also be valid for ponderable masses.

Let us now enumerate the properties of the electron's motion resulting from the system of equations (A) that are accessible to experiment.

1. From the second equation of the system (A) it follows that an electric force Y and a magnetic force N have an equally strong deflective effect on an electron moving with velocity v if $Y = Nv/V$. Thus we see that, using our theory, it is possible to determine the velocity of the electron from the ratio of the magnetic deflection A_m to the electric deflection A_e for arbitrary velocities, by applying the law

$$\frac{A_m}{A_e} = \frac{v}{V}.$$

This relation can be tested experimentally since the velocity of the electron can also be measured directly, e.g., using rapidly oscillating electric and magnetic fields.

2. From the derivation of the kinetic energy of an electron it follows that the potential difference traversed by the electron and the velocity v that the electron acquires must be related by the equation

$$P = \int X\,dx = \frac{\mu}{\epsilon}V^2 \left\{ \frac{1}{\sqrt{1-\left(\frac{v}{V}\right)^2}} - 1 \right\}.$$

3. We calculate the radius of curvature R of the path of the electron if a magnetic force N, acting perpendicularly to its velocity, is present (as the only deflecting force). From the second of equations (A) we obtain:

$$-\frac{d^2y}{dt^2} = \frac{v^2}{R} = \frac{\epsilon}{\mu}\frac{v}{V}N \cdot \sqrt{1-\left(\frac{v}{V}\right)^2}$$

or

$$R = V^2 \frac{\mu}{\epsilon} \frac{\frac{v}{V}}{\sqrt{1-\left(\frac{v}{V}\right)^2}} \cdot \frac{1}{N}.$$

These three relations are a complete expression of the laws by which, according to the theory presented here, the electron must move.

In conclusion, let me note that my friend and colleague M. Besso steadfastly stood by me in my work on the problem discussed here, and that I am indebted to him for several valuable suggestions.

(*Annalen der Physik* 17 [1905]: 891–921)

EDITORIAL NOTES

[1]In the 1913 reprint, a note was added after "be valid": "What is meant is, 'be valid in the first approximation.' " If Einstein did not write

the additional notes to this paper, the contents of some of the notes suggest that he was consulted.

[2]Einstein introduces the designations Ξ, H, Z for the coordinates for the x', y', x' axes of the moving system.

[3]In the 1913 reprint, the following note is appended to the end of this line: "The Lorentz transformation equations are more simply derivable directly from the condition that, as a consequence of these equations, the relation $\xi^2 + \eta^2 + \zeta^2 = V^2\tau^2 = 0$ shall have the other $x^2 + y^2 + z^2 - V^2t^2 = 0$ as a consequence."

[4]See previous note.

[5]This result later became known as "the clock paradox." In 1911, Langevin seems to have first introduced human travelers, leading to the alternate name, "the twin paradox."

[6]In the 1913 reprint, the following note is appended to the word "Unruhuhr": "In contrast to the 'pendulum clock,' which—from the physical standpoint—is a system, to which the earth belongs; this had to be excluded."

[7]This fraction should be $\frac{w_\eta}{w_\xi}$.

[8]The term "motional magnetic force" was introduced by Heaviside. Einstein later defined the "magnetomotive force" as the force acting on a unit of magnetic charge moving through an electric field. To the order of approximation used in the discussion of "electromotive force," the magnetomotive force is given by $-1/V\,[\mathbf{v}, \mathbf{E}]$, where $\mathbf{E} = (L, M, N)$, $\mathbf{v} = (v, 0, 0)$, and the bracket is a vector product.

[9]Corrected by Einstein in a reprint copy to "for $v = -V$, $v = -\infty$."

[10]In ibid., "the connecting line 'light source–observer' " was canceled and interlineated with "direction of motion."

[11]α should be φ.

[12]In a reprint copy, the denominator in the final term is corrected to "$1 - (v/V)^2$."

[13]In the 1913 reprint, the following note is appended to "call": "The definition of force given here is not advantageous as was first noted by M. Planck. It is instead appropriate to define force in such a way that the laws of momentum and energy conservation take the simplest form."