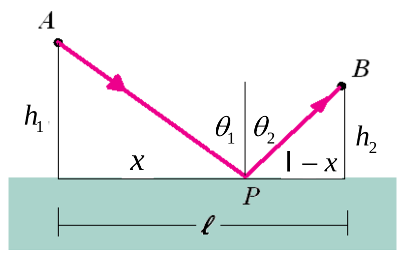
Fermat's Principle and the Laws of Reflection and Refraction

Fermat's principle states that "light travels between two points along the path that requires the least time, as compared to other nearby paths." From Fermat's principle, one can derive (a) the law of reflection [the angle of incidence is equal to the angle of reflection] and (b) the law of refraction [Snell's law]. This is problem 32-81 on page 864 of Giancoli. The derivations are given below.

Derivation of the laws of reflection and refraction

(a) Consider the light ray shown in the figure. A ray of light starting at point A reflects off the surface at point P before arriving at point B, a horizontal distance l from point A. We calculate the length of each path and divide the length by the speed of light to determine the time required for the light to travel between the two points.



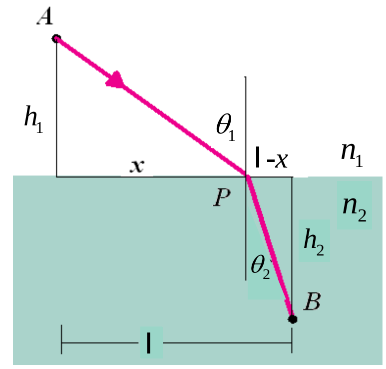
$$t = \frac{\sqrt{x^2 + h_1^2}}{c} + \frac{\sqrt{(l - x)^2 + h_2^2}}{c}$$

To minimize the time we set the derivative of the time with respect to *x* equal to zero. We also use the definition of the sine as opposite side over hypotenuse to relate the lengths to the angles of incidence and reflection.

$$0 = \frac{dt}{dx} = \frac{x}{c\sqrt{x^2 + h_1^2}} + \frac{-(l - x)}{c\sqrt{(l - x)^2 + h_2^2}} \quad \rightarrow$$

$$\frac{x}{\sqrt{x^2 + h_1^2}} = \frac{(l - x)}{\sqrt{(l - x)^2 + h_2^2}} \quad \rightarrow \quad \sin\theta_1 = \sin\theta_2 \quad \rightarrow \quad \boxed{\theta_1 = \theta_2}$$

(*b*) Now we consider a light ray traveling from point A to point B in media with different indices of refraction, as shown in the figure. The time to travel between the two points is the distance in each medium divided by the speed of light in that medium.



$$t=\frac{\sqrt{x^2+h_1^2}}{c/n_1}+\frac{\sqrt{(l-x)^2+h_2^2}}{c/n_2}$$

To minimize the time we set the derivative of the time with respect to *x* equal to zero. We also use the definition of the sine as opposite side over hypotenuse to relate the lengths to the angles of incidence and reflection.

$$0=\frac{dt}{dx}=\frac{n_1x}{c\sqrt{x^2+h_1^2}}+\frac{-n_2(l-x)}{c\sqrt{(l-x)^2+h_2^2}}\rightarrow\frac{n_1x}{\sqrt{x^2+h_1^2}}=\frac{n_2(l-x)}{\sqrt{(l-x)^2+h_2^2}}\rightarrow\boxed{n_1\sin\theta_1=n_2\sin\theta_2}$$