Differential Equations and Escape Velocity

by James D. Nickel

By definition, a differential equation is an equation that involves an unknown *function* (not an unknown *variable*) and its derivative or derivatives. For example, $\frac{dy}{dx} = 5x + 3$ is a differential equation involving the unknown function y. Hence, the solution to a differential equation is not a number; *it is a function or functions.*

Note: To understand this essay requires some basic knowledge of the differential and integral calculus and its applications to the physics of motion; i.e., the position, velocity, and acceleration functions.

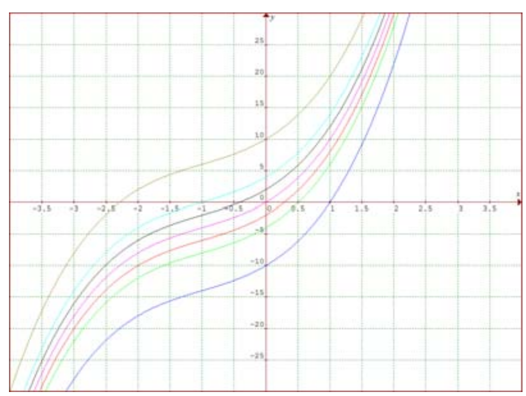
Differential equations are essential to scientific investigation where one needs to solve the relationship between rate of change (hence, derivatives) of continuously varying quantities. These types of situations appear in the disciplines of physics, chemistry, biology, engineering, and economics. Hence, for the Biblical Christian, the study of differential equations is a foundational dominion-related task.

Ordinary Differential Equations

A differential equation is *ordinary* if the unknown function depends on only one independent variable. If the unknown function depends upon two or more independent variables, the differential equation is called a *partial* differential equation. The mechanisms of solving partial differential equations are more complex than ordinary differential equation and that is why courses in differential equations start with the analysis of the "ordinary." The *order* of a differential equation is the order of the highest derivative appearing in the equation. Remember, you can calculate the first derivative of a function (first order), the second derivative of a function (second order), etc. This process ends when you finally get a constant function (e.g., f(x) = k).

Let's consider the function $f(x) = x^3 – 3x^2 + 6x – 10$. This function is a polynomial function and its first derivative is $f'(x) = 3x^2 – 6x + 6$. If we set y = f(x), then we write the first derivative is $\frac{dy}{dx} = 3x^2 – 6x + 6$. This equation is an example of an ordinary (first order) differential equation.

Differential Equations: General and Particular Solutions



Given the differential equation $\frac{dy}{dx} = 3x^2 – 6x + 6$, the *general* solution is $y = \int (3x^2 - 6x + 6)dx = x^3 – 3x^2 + 6x + C$ where C is an arbitrary constant (it can be any real or even a complex number). The graph at right shows some of those solutions; i.e., when C = 0, 2, -2, 4, -4, 10, and -10. These solutions comprise a *family of curves.*

If we know a specific coordinate (x-value, y-value), then we can generate the *particular* solution (i.e., find the specific C). Let say that we know that (1, 3) lies on the graph of the function of the solution. Using $y = x^3 – 3x^2 + 6x + C$ and substituting x = 1, y = 3, we get:

$$3 = 1 – 3 + 6 + C \Leftrightarrow C = -1$$

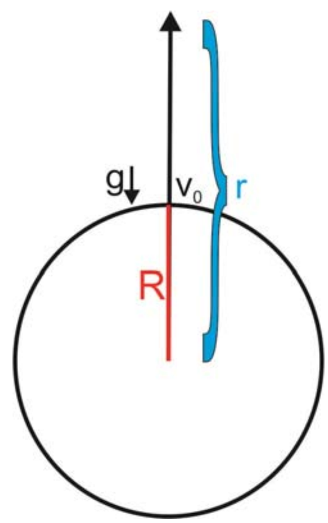
Hence, if we know the coordinate (1, 3), then $y = x^3 - 3x^2 + 6x - 1$ is a *particular* solution to all of these differential equations:

$\frac{dy}{dx} = 3x^2 - 6x + 6$ (first order)

$\frac{d^2y}{dx^2} = 6x - 6$ (second order)

$\frac{d^3y}{dx^3} = 6$ (third order)

The Question of Escape Velocity



Many problems in physics involve differential equations of the first order. In the rest of this essay, we will consider the problem of determining the velocity needed for an object, shot straight up (or projected in a radial direction outward from the earth), to *escape* the gravitational pull of the earth.

Since the object is shot upward in a radial direction, we shall assume that the pathway of the object takes place entirely on a line that, if continued toward the earth, intersects the center of the earth.

According to Newton's law of gravitation, the acceleration of the object will be inversely proportional to the square of the distance from the object to the center of the earth. Why? Since F = ma (Newton's Second law where *F* = force, *m* = mass, and *a* = acceleration), $F = \frac{GmM}{d^2}$ (Newton's law of gravitation where *m* and *M* and the mass of two objects, *d* is the distance between them and *G* is the gravitational constant of the system under consideration), then, by substitution (second law with the law of gravitation), $ma = \frac{GmM}{d^2} \Leftrightarrow a = \frac{GM}{d^2}$ (acceleration inversely proportional to the square of the distance).

Let *m* be the mass of the projected object and *M* be the mass of the earth. Let *r* = varying distance between the two objects and R = radius of the earth. If we let *t* = time, *v* = velocity of the object, *a* = acceleration (in calculus *a* is the first derivative of the velocity/time function or $a = \frac{dv}{dt}$), and *k* = constant of proportionality in Newton's law (we let k = GM), we get:

$$a = \frac{dv}{dt} = \frac{k}{r^2}$$

Note that the acceleration of the object has to be *negative* because its velocity will be *decreasing* (the pull of gravity will be slowing it down). Hence, the constant *k* is negative. When r = R, then a = -g (the acceleration of gravity at the surface of the earth), we get:

$$-g = \frac{k}{R^2}$$

Therefore, $k = -gR^2$. Substituting this result into $a = \frac{k}{r^2}$, we get:

$$a = -\frac{gR^2}{r^2}$$

Since $a = \frac{dv}{dt}$ and $v = \frac{dr}{dt}$ (velocity is the first derivative of the position function), we get:

$$a = \frac{dv}{dt} = \left(\frac{dr}{dt}\right)\left(\frac{dv}{dr}\right) = v\left(\frac{dv}{dr}\right)$$

We now have a differential equation for the velocity:

$$v\left(\frac{dv}{dr}\right) = -\frac{gR^2}{r^2}$$

By using a method of solution called "separation of variables," we first get (multiplying both sides by *dr*):

$$vdv = -gR^2\frac{dr}{r^2} \Leftrightarrow vdv = (-gR^2)r^{-2}dr$$

Integrating, we obtain the family of solutions:

$$\frac{v^2}{2} = -gR^2\frac{r^{-1}}{-1} + C \Leftrightarrow \frac{v^2}{2} = \frac{gR^2}{r} + C \Leftrightarrow v^2 = \frac{2gR^2}{r} + C$$

Let's assume that the object leaves the earth's surface with initial velocity = v_0. Hence, $v = v_0$ when $r = R$. From this, we can calculate C:

$$v_0^2 = \frac{2gR^2}{R} + C \Leftrightarrow v_0^2 = 2gR + C \Leftrightarrow C = v_0^2 - 2gR$$

Knowing C, we can now state an equation that governs the velocity, *v*, of an object projected in a radial direction outward from the earth's surface with an initial velocity v_0:

Velocity Equation:

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$$

Stunning Conclusions

We want to determine the velocity required for the object to escape the gravitational pull of the earth. In review, we know that at the surface of the earth, i.e., at $r = R$, the velocity is positive, i.e., $v = v_0$. Examining the right side of the *Velocity Equation* reveals that the velocity of the object will remain positive *if and only if*:

$$v_0^2 - 2gR \geq 0$$

If this inequality holds true, it means that the *Velocity Equation* will remain positive since it cannot vanish, is continuous over time, and is positive at r = R. When the inequality does *not* hold true, then, by implication, this inequality is true:

$$v_0^2 - 2gR < 0$$

Hence, there will be a *critical value* of *r* for which the right side of the *Velocity Equation* is zero. In other words, the object would stop, the velocity would change from positive to negative, *and the object would return, in free fall, to the earth.*

Our conclusion is that an object projected from the earth with a velocity v_0 such that:

$$v_0^2 - 2gR \geq 0 \Leftrightarrow v_0^2 \geq 2gR \Leftrightarrow v_0 \geq \sqrt{2gR}$$

will *escape* from the earth's gravitational pull. Hence, the minimum such velocity, i.e.,

Escape Velocity Equation:

$$v_0 = \sqrt{2gR}$$

is the *velocity of escape*.

Let's now calculate the value of the escape velocity for the earth. We approximate R = 3960. Near the surface of the earth, g = 32.16 feet per second every second or 32.16 ft/s^2. Converting to miles, we get g = 6.09 × 10^{-3} m/s^2. Substituting these values into the *Escape Velocity Equation*, we get:

$$v_0 = \sqrt{2gR} = \sqrt{2\left(\frac{6.09\times10^{-3}\text{m}}{\text{s}^2}\right)(3960\text{m})} = \sqrt{\frac{48.2328\text{m}^2}{\text{s}^2}} = 6.94 \text{ m/s (three significant figures)}$$

This is an excellent estimate but we must realize first that we are considering the object, following the lead of Galileo and Newton, as an "idealized" point (not a ballistic-type rocket). Second, we must note that wind resistance must be accounted for and it will require an adjustment to this escape velocity. Other mathematical methods, somewhat more complicated, are invoked to do this.

There is an amazing unity in diversity in this formula (reflecting the unique power of mathematics); i.e., it can be used to calculate the velocity required to escape from other objects (planets or moons) of our solar system, as long as *R* and *g* and given their correct values.

Another note of amazement is that with a few measurements provided by science (the radius of the earth and the pull of gravity near the surface of the earth), *one can quantify a rate that could not be determined in any other way.* In the 17th century, Newton was able to estimate the escape velocity well before the invention, in the 20th century, of the ballistic-type rocket. The initial velocity applied to such rockets (e.g., in the NASA moon program in the 1960s and 1970s) have confirmed that these objects do, indeed, escape the gravitational pull of the earth.

The derivation of the *Escape Velocity Equation* requires an understanding of the physics of motion (the result of the work in celestial mechanics by Galileo, Kepler and Newton in the 16th and 17th centuries) and the application of derivatives, integrals, the family of solutions to differential equations, and the order and syntax of the language of Algebra. Every new step is logically derived from the previous steps. The flow of this logic, to one mathematically attuned to it, borders on the sublime symmetry of poetry. The beauty revealed by such logical demonstrations is the primary reason why men and women are attracted to the discipline of mathematics and why many engage it as their lifetime vocation (either as research mathematicians or teachers of mathematics).

Scientists who deny that the covenant-keeping God of Scripture upholds by His omnipotence the rational and orderly movements of the universe arrive at an epistemological impasse when they consider *why* such logical derivations harmonize with "nature."

Dr. Remo Ruffini (1942-), Professor of Theoretical Physics at the University of Rome and President of the International Centre for Relativistic Astrophysics, makes an amazing concession as he considers the conundrum generated by the epistemological assumptions of modern science:

> "How a mathematical structure can correspond to nature is a mystery. One way out is just to say that the language in which nature speaks is the language of mathematics. This begs the question. Often we are both shocked and surprised by the correspondence between mathematics and nature, especially when the experiment confirms that our mathematical model describes nature perfectly."[1]

Later, Dr. Ruffini admitted that the mystery of mathematical effectiveness can be solved by positing the Biblical God. Suppressing this evident truth in unrighteousness (Romans 1), he rejected this *ultimate* explanation. According to Rousas J. Rushdoony, Ruffini, like a host of mathematicians and scientists of the modern world, "prefers to deny the theoretical possibility of a correlation and meaning than to admit the reality of the Creator God."[2]

May God increase the tribe of those who sense a calling of God directing them into these fields to salt them with the Truth.

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[1] Remo J. Ruffini, "The Princeton Galaxy," interviews by Florence Heltizer, *Intellectual Digest*, 3 (1973), p. 27.

[2] Rousas J. Rushdoony, *The Philosophy of the Christian Curriculum* (Vallecito: Ross House Books, 1981), p. 102.