Let's do the math: Escape Velocity

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Well, time to have a little bit of fun with math and physics. You did a simulation to try to figure out what the escape velocity would be, but we can avoid all that. We can actually come up with a formula that gives the escape velocity from any object that's circular, whatever its mass is. So right now we are going to come up with that formula.

Hmm. You know, we're already stuck. It's like, you want to come up with a formula, so where do you start. Well, there's a couple secrets to it. I'll tell you the first one.

The first one is to realize that if you jump off the ground and you don't come back, what that means is that, in terms of work or energy, is that you are looking for the amount of work it takes to jump and jump to infinity because that's what it means, really, to not come back. You are jumping to infinity. So the first part of what we are going to do is compute how much work it takes to jump to infinity. Then after that, I'll tell you the second secret we need in order to compute the escape velocity.

Work Formula

You remember Newton's famous formula

$F=ma$

Force=mass times acceleration

If we are near the surface of the earth, that becomes

$F=mg$

If you remember, g is about 9.8 meters per second squared, meaning, near the surface of the earth, any object accelerates toward the earth at an acceleration of 9.8 meters per second squared.

Work is Force times distance (s)

$W=Fs$

And if we plug in Force here, we get

$W=mgs$

Work equals mgs (m times g times s).

Let's take the mass of a cat. So that cat is, say, 5 kg, and g, as we said, is 9.8, and the distance we lift our cat off the ground, maybe its 2 meters. So, roughly speaking, this is 5 times 10 times 2, so this is about

20 joules of work. Unfortunately, 20 joules is about one fourth of a food calorie, which means that lifting your cat up once is not a good way to lose weight.

So this all well and good…unfortunately, it's not that simple. This formula only works near the surface of the earth. As I'm sure you know by now, as you get further and further away from the earth, the force of gravity changes, and so we have to take that into account.

How do we take that into account? Who comes to mind?

Yes, Sir Isaac Newton, as usual.

Gravity Formula

Isaac Newton thought of his force of gravity formula while listening to bad rap music on his iPod, as most of you know.

And here is his gravity formula.

$$F = \frac{GMm}{s^2}$$

F equals GMm (G times M times m) over s squared.

G is the gravitation constant that he came up with. It doesn't matter what the masses are of the objects. G remains the same. And if we go over here, I've already got G written down in these units.

$$G = 6.67 \ x \ 10^{-11} \ \frac{m^3}{kg \cdot sec^2}$$

And while I'm here, I've got the Mass of the Earth (M_E) written down and the Radius of the Earth (R_E) written down.

$$M_E = 5.98 \ x \ 10^{24} kg$$

$$R_E = 6.38 \ x \ 10^6$$

These are standard metric units of meters, kilograms, seconds, things like that.

Work Done Formula (Integral Calculus)

Ok, so here we have Newton's formula and we want to use it in work equals force times distance, but, as we said, the distance changes, and so the force changes, and so what we need to do then is a little bit of integral calculus.

$$W = \int_{R_E}^{\infty} \frac{GM_E \, M}{S^2} \, ds$$

Work equals Force, I'll put the integral sign in for now. So G M_E (G times M_E times M), we'll compute the escape velocity from the earth to begin with, so I'll put M_E here. This is your mass (M). Over s squared, the distance from the center of the earth to where you are at any moment in time. We are integrating with respect to the distance. And what about the limits of the integral? Well, we're starting at distance of R_E, the radius of the earth, and we want to know how much energy it takes to go from the radius of the earth to infinity, so we put a little infinity here. Everybody loves infinity. There's our work formula.

Alright. If you haven't had integral calculus, don't freak out too much because once I get through to the answer here, you'll see that the final formula, you only need a little of algebra to understand.

But I'm going to work through this anyway.

Ok, the first thing to note is that, with respect to integrating with s, G M_E and M are all constants so we can take them out of the integral.

$$W = GM_E\, M \int_{R_E}^{\infty} \frac{1}{s^2}\, ds$$

And so we have G M_E M integrating from the radius of the earth to infinity of 1 over s squared d s (d times s).

$$W = GM_E\, M \int_{R_E}^{\infty} \frac{1}{s^2}\, ds = GM_E\, M \left[-\frac{1}{s}\right]_{R_E}^{\infty}$$

And I'll just copy over those constants G M_E M. And the integral of 1 over s squared is just negative one over s. So we have a negative 1 over s and we evaluate at infinity and R_E, the radius of the earth.

How nice I am to separate those out for you.

$$W = GM_E\, M \int_{R_E}^{\infty} \frac{1}{s^2}\, ds = GM_E\, M \left[-\frac{1}{s}\right]_{R_E}^{\infty} = GM_E\, M \left[0 - \left(\frac{-1}{R_E}\right)\right]$$

And if we put infinity and negative 1 over infinity, you probably know that's zero. So we get G M_E M times zero minus, and then we plug in R_E here and we get negative 1 over R_E.

$$W = GM_E\, M \int_{R_E}^{\infty} \frac{1}{s^2}\, ds = GM_E\, M \left[-\frac{1}{s}\right]_{R_E}^{\infty} = GM_E\, M \left[0 - \left(\frac{-1}{R_E}\right)\right] = \frac{GM_E\, M}{R_E}$$

And that comes out to be G M_E M times 1 over R_E, which means over R_E. And that, if you recall from the beginning, is the amount of work done. So here it is the amount of work done to move an object from the surface of the earth to infinity.

Kinetic Energy Formula

So where do we go from here? Well, the second secret of computing escape velocity is to realize that this work or energy only comes, in our example, only comes from the kinetic energy of you jumping. So you start in a crouch. You jump up. Once you leave your feet, you have so much kinetic energy, that's, assuming the earth is all there is in the universe, that's all the energy that's there. So we can equate the kinetic energy to that work done.

Let's erase a little bit here. And we will do just that.

$$KE = \frac{1}{2} mv^2$$

So the kinetic energy, I'm sure you remember that formula, one half mv squared (m times v squared).

$$W = \frac{GM_E M}{s^2}$$

And then we have our work formula here. Work equals G Mass of the earth your mass (G times M_E times M) over the distance (s) squared.

$$KE = W$$

And as we just said, the kinetic energy is equal to the work.

$$\frac{1}{2} mv^2 = \frac{GM_E M}{R_E}$$

So plugging these in, we get one half m v squared (m times v squared) equals G M_E M, your mass, over R_E. And we want to solve for v here, which will give us our escape velocity.

Escape Velocity Formula

So look at this nice formula. One thing we notice is there is a single m here and a single m here (on both sides of the equation), so they cancel. What does that tell us? It tells us the escape velocity is independent of whatever the mass is. So whether you weigh 1 ounce or 1000 pounds, your escape velocity from the earth is going to be the same. And so that's why we can talk about the escape velocity from the earth or another planet and not have to refer to the mass that is trying to get away.

$$\frac{1}{2} v^2 = \frac{GM_E}{R_E}$$

So this gives us one half v squared equals G M_E over R_E.

$$v^2 = \frac{2GM_E}{R_E}$$

Just a little algebra here, right? Multiply both sides by 2. We get v squared equals 2 G M_E (2 times G times M_E) over R_E.

$$v = \sqrt{\frac{2GM_E}{R_E}}$$

And finally we take the square root of both sides. V equals the square root of 2 G M_E (2 times G times M_E) over R_E.

Ladies and gentlemen, this is our formula for the escape velocity from the earth.

Let’s get some numbers in here. If we plug in the values we have here. I’m not going to write them down. I’m sure you can plug them in. We get about 11,200 meters per second, which is roughly speaking 7 miles per second.

If you did the simulation, this is about the number that you came up with.

So get down, jump up. By the time you’re through jumping, if your velocity is 7 miles a second, you’re out of here!

Finally, notice in this formula, that if we put the mass of any object and the radius of that object in this formula, we can use this to compute the escape velocity from any circular object. And, as a matter of fact, in the quiz that accompanies this lesson, you will be doing just that. For instance, you’ll compute the escape velocity from the moon.

I’ve got a question for you. What happens if the escape velocity is the speed of light or greater? You say, “Well, wait a minute. Nothing can travel faster than the speed of light.” And you’re right. There is a special name for objects that have an escape velocity that’s the speed of light or greater. And I’ll bet a lot of you know what that is. It’s called a black hole. Am I going to talk about black holes now? No, but one of the links to the next lesson is about black holes. And I have a few links, so you can go off in the direction you are interested in, but if you are interested in black holes, click on that link.

Ladies and gentlemen, that was Escape Velocity Made Easy by Tim Farage. Alright, click on the next link and we’ll go from here.